Covariant formulation of wave-particle interaction in a transverse magnetic field

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We consider the interaction of relativistic charged particles with a perpendicular monochromatic wave of arbitrary amplitude in the presence of a static magnetic field. We introduce a covariant formulation based on a super-Hamiltonian and we study the energy-time phase plane. This formalism considerably simplifies the analysis of the problem and improves previous theoretical models. It is shown that the system dynamics for the monochromatic perturbation is quite different from the usual wave-packet perturbation: with the present formulation a finite stochastic web with three topologically distinct regions is observed.
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The motion of charged particles in a uniform magnetic field perturbed by an electrostatic wave is a fundamental problem in plasma physics and nonlinear Hamiltonian dynamics that has been studied by several authors [1–10]. In particular, it is relevant to the plasma heating by high-frequency waves through cyclotron resonances [1,2], to particle acceleration [3], and to the so-called paradox of the disappearance of the Landau damping of a plasma wave in a weak transverse magnetic field [4]. Recently, Chernikov *et al.* have studied the nonrelativistic interaction of a charged particle with an electrostatic wave packet propagating perpendicularly to the magnetic field, when the driving frequency is resonant with the cyclotron frequency [5]. The most striking feature of this system is the presence of a stochastic web spreading over the entire phase space, allowing particle acceleration to large energies, by a process analogous to Arnol'd diffusion [11]. The study of the relativistic version of the wave-packet problem by Longcope and Sudan [6] showed, however, that particles cannot diffuse to high energies because, unlike the nonrelativistic regime, the unperturbed Hamiltonian is no longer degenerate, leading to the appearance of Kolmogorov-Arnold-Moser (KAM) surfaces that confine the stochastic web to regions below a critical energy.

In this Brief Report, we propose a covariant formulation for the fully relativistic problem of particle acceleration by a monochromatic electrostatic wave in the presence of a perpendicular magnetic field. The corresponding super-Hamiltonian considerably simplifies the problem, and allows for the study of the charged particle dynamics with arbitrary velocities in the presence of waves with arbitrary amplitude. In our treatment, the space-momentum variables are formally equivalent to the energy-time variables and the particle dynamics can be studied directly on the energy-time phase plane, giving a clearer insight to particle acceleration processes. We apply this formalism to study the particularly important case of resonance between the cyclotron and the wave motion. We show that in this case a finite web region is formed with a much richer topological structure than that described in previous works [6] where the less realistic wave-packet perturbation of infinite spectral width was used. For low-energy regions we identify a doubly periodic web, while for moderate energies this doubly periodic web is distorted and is gradually replaced by a singly periodic web. Finally, for high-energy particles, we observe the presence of KAM tori, where a stochastic web cannot develop.

The usual relativistic Hamiltonian of a charged particle of mass m and charge q in the presence of an electrostatic wave and a static magnetic field is

$$H\equiv E=\{c^2[\vec{p}-q\vec{A}(\vec{r},t)]^2+m^2c^4\}^{1/2}+q\Phi(\vec{r}-\vec{v}_p t), \tag{1}$$

where $\vec{v}_p$ is the phase velocity of the wave and $\vec{A}(\vec{r},t)$ is the vector potential associated with the external magnetic field. With the Hamiltonian (1), the canonical equations of motion are relativistically invariant but are not covariant [12]. Alternatively, we can use a super-Hamiltonian, which cannot be identified with the particle energy E, but which is covariant in the Minkowski or relativistic space. Using the four-momentum $\hat{p}=(E/c,\vec{p})$, and the four-potential $\hat{A}=(\Phi/c,\vec{A})$ we write the four-kinetic momentum as $\hat{P}=\hat{p}-q\hat{A}$. The square module of $\hat{P}$ is $P^\alpha P_\alpha=-m^2c^2$, and we can define the super-Hamiltonian $\hat{H}$ as

$$\hat{H}(\hat{r},\hat{p})\equiv-\frac{1}{2}mc^2=\frac{\hat{P}^2}{2m}, \tag{2}$$

which generates the equations of motion for the new canonical variables in the Minkowski space: the four-position $\hat{r}=(ct,\vec{r})$, and the four-momentum $\hat{p}$. The proper time τ plays the usual role of t. From the definition of Eq. (2) it is obvious that the new Hamiltonian is covariant. We now consider the dynamics of a charged particle, in the laboratory frame, in the presence of a static magnetic field $\vec{B}=B_0\vec{e}_y$ and a perpendicular monochromatic electrostatic wave of frequency ω, wave number k, and amplitude E_0, described by the electric field $\vec{E}=E_0\sin(kz-\omega t)\vec{e}_z$. In the Coulomb gauge, the explicit expression of the four-potential is $\hat{A}=(-(E_0/ck)\sin[kr^3-(\omega/c)r^0],B_0r^3,0,0)$, and the super-Hamiltonian is given by

$$h=\frac{p^2}{2}-\frac{u^2}{2}+\left(\frac{\Omega}{N\omega}\right)^2\frac{q^2}{2}-\alpha u\sin(q-y/N)$$
$$+\frac{\alpha^2}{4}\cos(2q-2y/N)-\frac{\alpha^2}{4}$$
$$=-\frac{1}{2}, \tag{3}$$

where the canonical variables (p,q) are the normalized momentum $p=p_3/mc=p_z/mc$, and the normalized position $q=kr^3=z\omega N/c$. The canonical variables (u,y) are the normalized energy $u=p_0/mc=-E/mc^2$, and the normalized time $y=kr^0=\omega Nt$. In Eq. (3) , Ω is the nonrelativistic cyclotron frequency, $N=ck/\omega$ is the refraction index of the electrostatic wave, and $\alpha=qE_0c/(\omega Nmc^2)$ is the normalized amplitude of the perturbation. The super-Hamiltonian is independent of $r^1=x$ and $r^2=y$, hence $p_1=p_x$ and $p_2=p_y$ are constants of the motion. Without loss of generality, these constants are set to be zero in Eq. (3). We observe that the unperturbed Hamiltonian $(\alpha=0)$ is degenerate in the space-momentum variables and nondegenerate in the energy-time canonical variables. The space-momentum degeneracy was already observed in the nonrelativistic case [8]. The permanence of a stochastic web in the relativistic description is a reminiscence of this fact, while the presence of KAM tori is due to the global nondegeneracy of the system, induced by the strong nonlinearity in the energy-time dynamics. This duality can be easily explored in this covariant formalism since both space momentum and time energy have equivalent formal meanings, i.e., pairs of canonical variables in Minkowski space. Moreover, the explicit expression of the covariant Hamiltonian is much easier to handle than Eq. (1), and a nonlinear analysis, similar to that used in the nonrelativistic domain, can be performed.

We now generalize the procedure outlined by Zaslavskii *et al.* [8] to this two-dimensional (2D) Hamiltonian. We first transform the variables (p,q) into the corresponding zero-order action-angle variables (I_1,θ_1), where $p=(2I_1\Omega/N\omega)^{1/2}\cos\theta_1$, and $q=(2N\omega I_1/\Omega)^{1/2}\sin\theta_1$. Instead of considering the system dynamics in the Poincaré surface of section (I_1,θ_1), we focus our attention on the phase plane (E,t), which can give us direct insight on the charged particle acceleration. For that purpose, we apply the formulation of Zaslavskii *et al.* to the canonical variables u (energy) and y (time), using a F_2-type generating function (Goldstein notation) given by $F_2=\tilde{I}_2(\theta_1-y\Omega/N\omega)+\tilde{I}_1\theta_1$, corresponding to the transformation of variables: $\tilde{I}_2=-uN\omega/\Omega$, $\tilde{\theta}_2=\theta_1-y\Omega/N\omega$, $I_1=I_1-\tilde{I}_2$, and $\tilde{\theta}_1=\theta_1$. Using the standard properties of the Bessel functions it is possible to write the new Hamiltonian $\tilde{H}$ as a function of the new canonical variables $(\tilde{I}_1,\tilde{\theta}_1,\tilde{I}_2,\tilde{\theta}_2)$:

$$\tilde{H}=H_0(\tilde{I}_1,\tilde{I}_2)+\alpha\frac{\Omega}{N\omega}\tilde{I}_2\sum_n J_n(\tilde{r})\sin\left(n\tilde{\theta}_1-\frac{\omega}{\Omega}(\tilde{\theta}_1-\tilde{\theta}_2)\right)$$
$$-\frac{\alpha^2}{4}\sum_m J_m(2\tilde{r})\sin\left(m\tilde{\theta}_1-\frac{\pi}{2}-2\frac{\omega}{\Omega}(\tilde{\theta}_1-\tilde{\theta}_2)\right), \tag{4}$$

where $H_0(\tilde{I}_1,\tilde{I}_2)=(\tilde{I}_1+\tilde{I}_2)\Omega/N\omega-\tilde{I}_2^2\Omega^2/2N^2\omega^2$ and $\tilde{r}=[2(\tilde{I}_1-\tilde{I}_2)N\omega/\Omega]^{1/2}$. We now proceed by considering the special case of nonrelativistic resonance, i.e., $\omega=n_0\Omega$. In this case, we can single out the resonant Hamiltonian h_R, which is independent of $\tilde{\theta}_1$:

$$h_R=H_0(\tilde{I}_1,\tilde{I}_2)+\frac{\alpha^2}{4}J_{2n_0}(2\tilde{r})\sin\left(\frac{\pi}{2}+2n_0\tilde{\theta}_2\right)$$
$$-\frac{\alpha}{n_0N}\tilde{I}_2J_{n_0}(\tilde{r})\sin(n_0\tilde{\theta}_2+\pi). \tag{5}$$

This Hamiltonian describes the resonant motion in the phase plane $(\tilde{I}_2,\tilde{\theta}_2)$-(energy, time), corresponding to the surface of section $\tilde{\theta}_1=\text{const}$. The second term of Eq. (5) describes the usual nonrelativistic web, while the third term gives the relativistic correction, which has a different periodicity and, as expected, is energy dependent. The frequency detuning $\delta\omega=\dot{\tilde{\theta}}_2$ for $\alpha=0$ is defined as $\delta\omega=\partial h_R(\alpha=0)/\partial\tilde{I}_2=1/Nn_0-\tilde{I}_2/N^2n_0^2$. Since $\delta\omega\neq 0$, the degeneracy is not present and the Hamiltonian h_R has no web. However, the structure of h_R is very close to a degenerate Hamiltonian ($\delta\omega\simeq 0$, if $\tilde{I}_2\simeq Nn_0$—nonrelativistic energies), and if the perturbation is strong enough, the contribution of the nonresonant terms in Eq. (4) leads to the appearance of a stochastic layer in the separatrix region that covers the gaps between different separatrices, resulting in a large stochastic web. The contribution of these nonresonant terms will be discussed below.

We now consider the topological properties of h_R. The fixed points $(\tilde{I}_{20},\tilde{\theta}_{20})$ of h_R verify $\partial h_R/\partial\tilde{I}_2=0$, and $\partial h_R/\partial\tilde{\theta}_2=0$:

$$-\alpha n_0N\sin(n_0\tilde{\theta}_{20})\cos(n_0\tilde{\theta}_{20})=\tilde{I}_{20}\frac{J_{n_0}(\tilde{r}_0)}{J_{2n_0}(2\tilde{r}_0)}\cos(n_0\tilde{\theta}_{20}+\pi), \tag{6}$$

$$\frac{1}{Nn_0}-\frac{\tilde{I}_{20}}{N^2n_0^2}-\frac{\partial}{\partial\tilde{I}_2}\left(\frac{\alpha}{Nn_0}\tilde{I}_2J_{n_0}(\tilde{r})\sin(n_0\tilde{\theta}_{20}+\pi)\right)_{\tilde{I}_2=\tilde{I}_{20}}$$
$$=-\frac{\alpha^2}{4}\sin\left(\frac{\pi}{2}+2n_0\tilde{\theta}_{20}\right)\frac{\partial J_{2n_0}(2\tilde{r})}{\partial\tilde{I}_2}\Bigg|_{\tilde{I}_2=\tilde{I}_{20}}. \tag{7}$$

The analysis of Eqs. (5)–(7) shows that four topological distinct regions exist, corresponding to the existence or nonexistence of different kinds of fixed points. For low energies, $\tilde{I}_2\ll\alpha n_0N/4$, the fixed points of h_R are located at $\partial J_{2n_0}(2\tilde{r}_0)/\partial\tilde{r}=0\wedge\cos(2n_0\tilde{\theta}_2)=0$ (elliptic fixed points) and $J_{2n_0}(2\tilde{r}_0)=0\wedge\sin(2n_0\tilde{\theta}_2)=0$ (hyperbolic fixed points). The width of the cells is simply given by $\Delta_{\text{cell}}=2(\tilde{r}_{sx}-\tilde{r}_0)=2\pi$, where $\tilde{r}_{\text{sx}}$ is the value of $\tilde{r}$ over the separatrix joining two hyperbolic points. It is straightforward to show that the periodicity of the web in this region is π/n_0 (Fig. 1). The stability analysis of this set of fixed points shows that the doubly periodic web only exists if

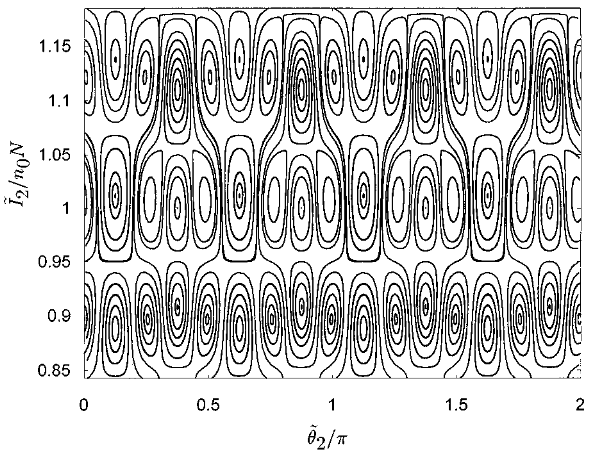

FIG. 1. Phase space $\mathcal{E}_{\tilde{I}_2,\tilde{\theta}_2}$ of the resonant Hamiltonian h_R in the doubly periodic region, for the parameters $n_0=4$, $N=5$, and $\alpha=8$, with $\tilde{I}_{10}=0$.

$$\tilde{I}_{20}\leqslant \alpha n_0 N/\sqrt{2}. \tag{8}$$

However, and before the disappearance of the doubly periodic fixed points, the relativistic corrections become important, and a transition region is present for energies between $\tilde{I}_2=\alpha n_0 N/4$ and the threshold of Eq. (8) (Fig. 2). In this region, the doubly periodic structure is mixed with a singly periodic web due to the third term of Eq. (5). For energies above the threshold of Eq. (8), the third term of Eq. (5) is dominant, the doubly periodic structure disappears, and the structure of the fixed points is drastically changed. In fact, for these energies, and assuming $N\gg 1$, the fixed points are located at $\partial J_{n_0}(\tilde{r}_0)/\partial\tilde{r}=0\wedge\cos(n_0\tilde{\theta}_2)=0$ (elliptic fixed points) and $J_{n_0}(\tilde{r}_0)=0\wedge\sin(n_0\tilde{\theta}_2)=0$ (hyperbolic fixed points). Unlike the first region, the periodicity of the web is now $2\pi/n_0$ (Fig. 3). Furthermore, the width of each web cell is, in this case, $\Delta_{\rm cell}\simeq 4\pi$. This means that for the weakly relativistic regime the acceleration process inside each web cell is more important than in the nonrelativistic regime. For high energies, the weblike structure becomes increasingly

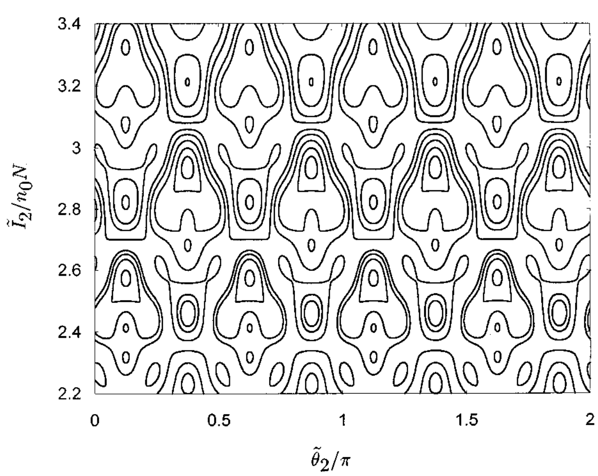

FIG. 2. Phase space $\mathcal{E}_{\tilde{I}_2,\tilde{\theta}_2}$ of the resonant Hamiltonian h_R in the transition region from the doubly periodic to the singly periodic weblike structure, for the same parameters of Fig. 1.

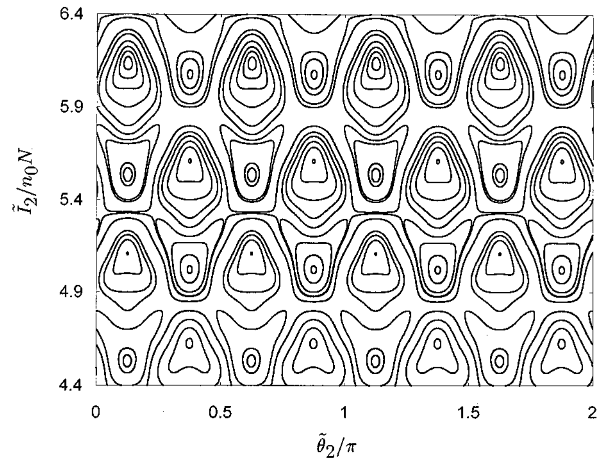

FIG. 3. Phase space $\mathcal{E}_{\tilde{I}_2,\tilde{\theta}_2}$ of the resonant Hamiltonian h_R in the region of the singly periodic web, with the same set of parameters of Fig. 1.

deformed giving rise to resonances similar to those of an asymmetric pendulum [13], and finally the resonance structure disappears, giving rise to KAM tori (Fig. 4). The threshold for this transition can be derived from the analysis of Eqs. (6) and (7), corresponding to the existence of the singly periodic fixed points, and it is given by

$$\tilde{r}^{3/2}=\alpha n_0^2 N^2\sqrt{2/\pi}. \tag{9}$$

Above this threshold the weblike structure does not exist, and stochastic acceleration along the web is no longer possible. This is equivalent to the stochastic web limit already observed for weak wave perturbations [6]. This intricate web structure was not identified in previous works because it was always considered a wide wave packet perturbation characterized by a single spatial period, equivalent to periodic δ kicks. The periodicity of the Dirac δ function is the same as the periodicity of the square of the δ function and, conse-

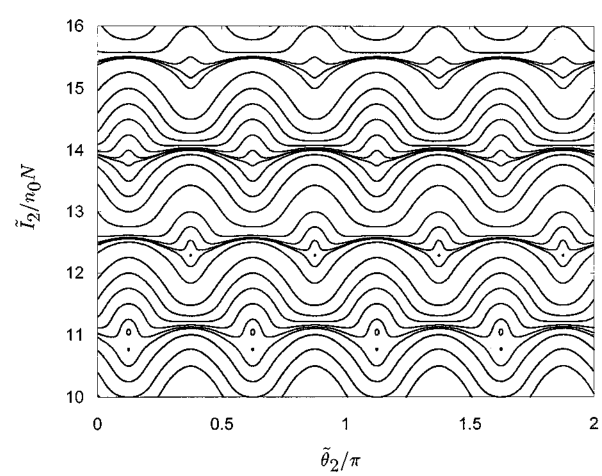

FIG. 4. Phase space $\mathcal{E}_{\tilde{I}_2,\tilde{\theta}_2}$, of the resonant Hamiltonian h_R, showing the transition from the singly periodic web to the KAM tori region. The analytical threshold of Eq. (9), giving the upper limit of the weblike region, is $\tilde{I}_2/n_0N=13.6$, for the parameters $n_0=4$, $N=5$, and $\alpha=8$, with $\tilde{I}_{10}=30Nn_0$.

quently, the corresponding resonant Hamiltonian does not show a web with different periodicities in the time variable.

The nonresonant terms of the super-Hamiltonian, which are the $\tilde{\theta}_1$-dependent terms in Eq. (4), perturb the motion near the separatrix and a stochastic layer will be formed. For the super-Hamiltonian in Eq. (5) the separatrix width can be estimated as usual, by calculating the Melnikov-Arnol'd integral [8,14]. In the doubly periodic region, the thickness of this layer is $\Delta(\tilde{r}_0)\sim(\sqrt{\tilde{r}_0}/\alpha^2)\exp(-\sqrt{\pi}\tilde{r}_0^{3/2}/\alpha^2 n_0^2 N^2)$. Similar calculations show that the thickness of the singly periodic web verifies $\Delta(\tilde{r}_0)\sim(\sqrt{\tilde{r}_0}n_0 N/\alpha\tilde{I}_{20})\exp[-\sqrt{\pi}(\tilde{r}_0/2)^{3/2}/\alpha\tilde{I}_{20}n_0 N]$. The stochastic region around each separatrix will overlap with similar regions from neighboring separatrices leading to the presence of a stochastic web. It should be pointed out that the dependence of the separatrix width on the amplitude of the perturbation ϵ is always of the form $\exp(-\mathrm{const}/\epsilon)$, as long as $N\gg 1$, in contrast with the $\exp(-\mathrm{const}/\sqrt{\epsilon})$ dependence of the Arnol'd web [8,14,15]. Moreover, since the amplitude of the perturbation is considerably different in the two web regions, it is expected that the qualitative behavior of the separatrix width will also be different in the two regions. A more detailed analysis will be considered in a future publication, in connection with the study of the motion of relativistic charged particles in a uniform magnetic field perturbed by a finite set of perpendicular electrostatic waves.

In summary, we have introduced a covariant formulation that allows for a more explicit description of the motion of relativistic charged particles in the presence of a static magnetic field and a perpendicular electrostatic wave. Using this formulation we have studied the phase plane time-energy, which can give direct information about the most efficient regimes for regular and stochastic particle acceleration, and can be generalized to other types of perturbation, namely, a finite set of perpendicular waves or obliquely propagating electrostatic waves. We have identified four topologically distinct phase-space regions and we have determined the boundaries of each region. The webs existing in the low-energy region are equivalent to those already identified in the nonrelativistic regime [5]. However, it has been shown that for moderate energies, and just before the transition to KAM tori, a singly periodic region is also present.

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